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ON THE GENERALIZED WRIGHT FUNCTION

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Dedicated to 60th birthday of Prof. F. Mainardi

Abstract

The paper is devoted to the study of the generalized Wright function. Conditions for the existence of ${}_p\Psi_q(z)$ are proved. The representations of the function in terms of the Mellin-Barnes integral and of the H-function are established. Special cases, involving the Mittag-Leffler function and its generalizations, are considered. The obtained results imply more precisely the known results.

Key words: Wright and generalized Wright functions, H-function, Mittag-Leffler-type functions
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1. Introduction

The paper deals with the generalized Wright function defined for $z \in \mathbb{C}$, $a_i, b_j \in \mathbb{C}$ and $\alpha_i,\ \beta_j \in \mathbb{R} = (-\infty, \infty)$ $(\alpha_i, \beta_j \neq 0;\ i = 1, 2, \cdots, p;\ j = 1, 2, \cdots, q)$ by the series

$$ {}_p\Psi_q(z) \equiv {}_p\Psi_q \left[\begin{array}{c|} (a_i, \alpha_i)_{1,p} \\ \\ (b_i, \beta_i)_{1,q} \end{array}\ z \right] = \sum_{k=0}^{\infty} \frac{\prod\limits_{i=1}^{p} \Gamma(a_i + \alpha_i k)}{\prod\limits_{j=1}^{q} \Gamma(b_j + \beta_j k)} \frac{z^k}{k!}, \tag{1.1} $$

where $\mathbb{C}$ is the set of complex numbers and $\Gamma(z)$ is the Euler gamma-function [8, Section 1.1]. The function in (1.1) was introduced by Wright [36] and is called the generalized Wright function, see [8, Section 4.1]. This function generalizes many special functions. Earlier in [34] Wright introduced the special case of the function (1.1) in the form

$$ \phi(\alpha, \beta; z) \equiv {}_0\Psi_1 \left[\begin{array}{c|} \text{---} \\ \\ (\beta, \alpha) \end{array}\ z \right] = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}. \tag{1.2} $$

with $z, \beta \in \mathbb{C}$ and $\alpha \in \mathbb{R}$, known as the Wright function [9, Section 18.1]. When $\alpha = \mu$, $\beta = \nu + 1$ and z is replaced by $-z$, the function $\phi(\mu, \nu + 1; -z)$ is denoted by $J_\nu^\mu(z)$:

$$J_\nu^\mu(z) \equiv \phi(\mu, \nu + 1; -z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\mu k + \nu + 1)} \frac{(-z)^k}{k!}, \tag{1.3}$$

and such a function is known as the Bessel-Maitland function, or the Wright generalized Bessel function, see [16, p. 352] and [25, (8.3)]. Some other particular cases of the generalized Wright function (1.1) are presented in Section 6. Wright in [35], [39] investigated by the "steepest descent" method the asymptotic expansions of the function $\phi(\alpha, \beta; z)$ for large values of z in the cases $\alpha > 0$ and $-1 < \alpha < 0$, respectively. In [35] he indicated the application of the obtained results to the asymptotic theory of partitions. In [36]-[38] Wright extended the last results to the generalized Wright function (1.1) and proved six theorems on the asymptotic expansion of ${}_p\Psi_q(z)$ for all values of the argument z under the condition

$$\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1. \tag{1.4}$$

The properties of the Wright function (1.2) were studied in a series of papers. Some results can be found in [9, Section 18.1] including the connection of $\phi(\alpha, \beta; z)$ with the Mittag-Leffler function (6.1) via the Laplace transform. We also indicate that some fractional integral relations for the function (1.2) were presented in [4], asymptotic relations for zeros of the Wright function $\phi(\alpha, \beta; z)$ were established in [19], and distributions of these zeros were investigated in [20]. Applications of the Wright function (1.2) to the operational calculus were given in [27], while to integral transforms of Hankel type - in [10] and [33]. A series of papers were devoted to the application of the Wright function $\phi(\alpha, \beta; z)$ in partial differential equations of fractional order generalizing the classical diffusion and wave equations. The solution of the boundary value problems for the corresponding fractional diffusion-wave equation were given by the fractional Green function involving the Wright function. Mainardi [22] probably first obtained such a result for the fractional diffusion-wave equation; see [28, Section 4.1.2]. One may find the results in this field in a survey paper [23]; see also [24]. In [3] and [21] the scale-invariant solutions of some partial differential equations of fractional order were also given in terms of some special cases of the generalized Wright function (1.1). In this paper we study the properties of the generalized Wright function (1.1). We establish the conditions for the existence of ${}_p\Psi_q(z)$ and prove its integral representations in terms of the Mellin-Barnes integral [8, Section 1.19] in the form:

$$
{}_p\Psi_q\left[\begin{array}{c|c} (a_i,\alpha_i)_{1,p} & \\ & z \\ (b_i,\beta_i)_{1,q} & \end{array}\right] = \frac{1}{2\pi i}\int_L \frac{\Gamma(s)\prod\limits_{i=1}^{p}\Gamma(a_i-\alpha_i s)}{\prod\limits_{j=1}^{q}\Gamma(b_j-\beta_j s)}(-z)^{-s}ds \tag{1.5}
$$

with a special path of integration L. This representation is used to represent the generalized Wright function (1.1) as the H-function, the theory of which can be found in [2], [8, Section 1.19], [14, Chapters 1 and 2], [26, Chapter 2], [30, Section 8.3] and [32, Chapter 1]. The paper is organized as follows. The conditions for the existence of ${}_p\Psi_q(z)$ are discussed in Section 2. The Mellin-Barnes integral representations of this function are proved in Sections 3 and 4. The latter results are applied in Section 5 to represent the generalized Wright function (1.1) as the H-function. In Sections 3-5 the corresponding results for the Wright function (1.2) and for the Bessel-Maitland function (1.3) are also given. Some other special cases of the generalized Wright function (1.1) involving the Mittag-Leffler function and some of its generalizations are presented in Section 6.

2. Existence of the generalized Wright function

In this section we give the conditions for the existence of the generalized Wright function ${}_p\Psi_q(z)$ in (1.1). To formulate the result we use the following notation:

$$
\Delta = \sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i, \tag{2.1}
$$

$$
\delta = \prod_{i=1}^{p}|\alpha_i|^{-\alpha_i}\prod_{j=1}^{q}|\beta_j|^{\beta_j}, \tag{2.2}
$$

$$
\mu = \sum_{j=1}^{q}b_j - \sum_{i=1}^{p}a_i + \frac{p-q}{2}. \tag{2.3}
$$

THEOREM 1. *Let $a_i, b_j \in \mathbb{C}$ and $\alpha_i,\ \beta_j \in \mathbb{R}$ $(i=1,2,\cdots,p;\ j=1,2,\cdots,q)$.*

(a) *If $\Delta > -1$, then the series in* (1.1) *is absolutely convergent for all $z \in \mathbb{C}$.*

(b) *If $\Delta = -1$, then the series in* (1.1) *is absolutely convergent for all values of $|z| < \delta$ and of $|z| = \delta$, $\Re(\mu) > 1/2$.*

P r o o f. (1.1) is the power series of the form

$$
{}_p\Psi_q\left[\begin{array}{c|c} (a_i,\alpha_i)_{1,p} & \\ & z \\ (b_i,\beta_i)_{1,q} & \end{array}\right] = \sum_{k=0}^{\infty} c_k z^k, \tag{2.4}
$$

where

$$c_k = \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k)} \frac{1}{k!} \quad (k \in \mathbf{N}_0). \tag{2.5}$$

We investigate asymptotic behavior of c_k when $k \to \infty$. In accordance with the Stirling formula for the Gamma-function [8, 1.18(2)]

$$\Gamma(z) = (2\pi)^{1/2} z^{z-1/2} e^{-z} \left[1 + O\left(\frac{1}{z}\right)\right] \quad (z \to \infty), \tag{2.6}$$

we have, as $k \to +\infty$,

$$\Gamma(a_i + \alpha_i k) \sim A_i \left(\frac{k}{e}\right)^{\alpha_i k} \alpha_i^{\alpha_i k} k^{a_i - 1/2}, \quad A_i = (2\pi)^{1/2} \alpha_i^{a_i - 1/2} e^{-a_i}, \tag{2.7}$$

for $i = 1, \cdots, p$, and

$$\Gamma(b_i + \beta_j k) \sim B_j \left(\frac{k}{e}\right)^{\beta_j k} \beta_j^{\beta_j k} k^{b_j - 1/2}, \quad B_j = (2\pi)^{1/2} \beta_j^{b_j - 1/2} e^{-b_j}, \tag{2.8}$$

for $j = 1, \cdots, q$, while

$$k! \sim (2\pi)^{1/2} \left(\frac{k}{e}\right)^{k} k^{1/2} \quad (k \to +\infty). \tag{2.9}$$

Using the estimates (2.7)-(2.9) and taking (2.5) into account, we obtain the estimate for c_k, as $k \to +\infty$, of the form:

$$c_k \sim A \left(\frac{k}{e}\right)^{-(\Delta+1)k} \left(\prod_{i=1}^{p} \alpha_i^{\alpha_i} \prod_{j=1}^{q} \beta_j^{-\beta_j}\right)^{k} k^{-[\mu+1/2]} \quad (k \to \infty), \tag{2.10}$$

where

$$A = (2\pi)^{(p-q-1)/2} \frac{\prod_{i=1}^{p} \left[\alpha_i^{a_i - 1/2} e^{-a_i}\right]}{\prod_{j=1}^{q} \left[\beta_j^{b_j - 1/2} e^{-b_j}\right]}. \tag{2.11}$$

Now the results in (a) and (b) follow from the known convergence principles for the power series in (2.4), which completes the proof of theorem.

COROLLARY 1.1. *Let* $a_i, b_j \in \mathbb{C}$ *and* $\alpha_i, \beta_j \in \mathbb{R}$ $(i = 1, 2, \cdots, p;\ j = 1, 2, \cdots, q)$ *be such that the condition in* (1.4) *is satisfied. Then the generalized Wright function* ${}_p\Psi_q(z)$ *is an entire function of* z.

Corollary 1.2. *Let* $\alpha \in \mathbb{R}$ *be and* $\beta \in \mathbb{C}$.

(a) *If* $\alpha > -1$, *then the series in* (1.2) *is absolutely convergent for all* $z \in \mathbb{C}$.

(b) *If* $\alpha = -1$, *then the series in* (1.2) *is absolutely convergent for all values of* $|z| < 1$ *and of* $|z| = 1$, $\Re(\beta) > 1$.

Corollary 1.3. *If* $\alpha > -1$ *and* $\beta \in \mathbb{C}$, *then the Wright function* $\phi(\alpha, \beta; z)$ *is an entire function of* z.

Corollary 1.4. *If* $\mu > -1$ *and* $\nu \in \mathbb{C}$, *then the Bessel-Maitland function* $J_\nu^\mu(z)$ *is an entire function of* z.

3. The representation of the generalized Wright function by the Mellin-Barnes integral

In this section we give the conditions for the generalized Wright function ${}_p\Psi_q(z)$ in (1.1) to be represented by the Mellin-Barnes integral of the form (1.5) in the case when α_i $(1 \le i \le p)$ and β_j $(1 \le j \le q)$ are positive numbers and $L = L_{-\infty}$ is a loop situated in a horizontal strip starting at the point $-\infty + i\varphi_1$ and terminating at the point $-\infty + i\varphi_2$ with $-\infty < \varphi_1 < \varphi_2 < \infty$.

Let $s \in \mathbb{C}$. We suppose that $a_i, b_j \in \mathbb{C}$ and $\alpha_i > 0$, $\beta_j > 0$ $(i = 1, 2, \cdots, p;\ j = 1, 2, \cdots, q)$ be such that poles

$$b_l = -l \quad (l = 0, 1, 2, \cdots) \tag{3.1}$$

of the Gamma function $\Gamma(s)$ and poles

$$a_{ik} = \frac{a_i + k}{\alpha_i} \quad (i = 1, 2, \cdots, p;\ k = 0, 1, 2, \cdots) \tag{3.2}$$

of the Gamma functions $\Gamma(a_i - \alpha_i s)$ $(i = 1, 2, \cdots, p)$ do not coincide:

$$\alpha_i l + a_i + k \neq 0 \quad (i = 1, 2, \cdots, p;\ l, k = 0, 1, 2, \cdots). \tag{3.3}$$

We also suppose that poles a_{ik} in (3.2) are simple:

$$(a_i + k)\alpha_j \neq (a_j + l)\alpha_i \quad (i \neq j;\ i, j = 1, 2, \cdots, p). \tag{3.4}$$

There holds the following result, in which Δ, δ and μ are given by (2.1), (2.2) and (2.3), respectively.

Theorem 2. *Let* $a_i, b_j \in \mathbb{C}$ *and* $\alpha_i > 0$, $\beta_j > 0$ $(i = 1, 2, \cdots, p;\ j = 1, 2, \cdots, q)$ *be such that the conditions in* (3.3) *and* (3.4) *are satisfied. Let either of the following conditions holds*:

$$\Delta > -1, \quad z \neq 0; \tag{3.5}$$

$$\Delta = -1, \quad 0 < |z| < \delta; \tag{3.6}$$

$$\Delta = -1, \quad |z| = \delta, \quad \Re(\mu) > \frac{1}{2}. \tag{3.7}$$

Then the generalized Wright function ${}_p\Psi_q(z)$ *in* (1.1) *has the integral representation by the Mellin-Barnes integral of the form* (1.5) :

$${}_p\Psi_q\left[\begin{array}{c|c} (a_i,\alpha_i)_{1,p} & \\ & z \\ (b_i,\beta_i)_{1,q} & \end{array}\right] = \frac{1}{2\pi i}\int_{L_{-\infty}} \frac{\Gamma(s)\prod_{i=1}^{p}\Gamma(a_i-\alpha_i s)}{\prod_{j=1}^{q}\Gamma(b_j-\beta_j s)}(-z)^{-s}ds, \tag{3.8}$$

where the path of integration $L_{-\infty}$ *separates all poles* b_l *in* (3.1) *to the left and all poles* a_{ik} *in* (3.2) *to the right.*

P r o o f. Let

$$I(z) = \frac{1}{2\pi i}\int_{L_{-\infty}} \mathcal{H}^{1,p}_{p,q+1}(s)(-z)^{-s}ds, \tag{3.9}$$

where

$$\mathcal{H}^{1,p}_{p,q+1}(s) = \frac{\Gamma(s)\prod_{i=1}^{p}\Gamma(a_i-\alpha_i s)}{\prod_{j=1}^{q}\Gamma(b_j-\beta_j s)}. \tag{3.10}$$

As a particular case of (2.6), there holds the asymptotic formula at infinity

$$\Gamma(x+iy) \sim (2\pi)^{1/2}|y|^{x-1/2}e^{-x-\pi|y|/2} \quad (x,y\in\mathbb{R};\ |y|\to\infty). \tag{3.11}$$

By applying this relation, direct calculations yield the following asymptotic estimate for the function in (3.10):

$$\mathcal{H}^{1,p}_{p,q+1}(\sigma+it) \sim B\left(\frac{e}{|t|}\right)^{(\Delta+1)|t|}\delta^{-|t|}|t|^{-[\mathrm{Re}(\mu)+1/2]} \quad (\sigma,t\in\mathbb{R};\ t\to-\infty), \tag{3.12}$$

where

$$B = (2\pi)^{(p+1-q)/2}e^{q-p-\sigma\pi}\frac{\prod_{j=1}^{q}\left[\beta_j^{-\Re(b_j)+1/2}e^{\Re(b_j)-1}\right]}{\prod_{i=1}^{p}\left[\alpha_i^{-\Re(a_i)+1/2}e^{\Re(a_i)-1}\right]} \tag{3.13}$$

and Δ, δ and μ are given (2.1),(2.2) and (2.3).

Since

$$(-z)^{-(t+i\sigma)} = |z|^{-t}e^{\sigma\arg(-z)}e^{-i[t\arg(-z)+\sigma\log(|z|)]}, \tag{3.14}$$

then, in accordance with the known convergence principles for the improper integrals, the integral in (3.9) is convergent, provided that either of the conditions in (3.5), (3.6) and (3.7) is satisfied.

To prove the relation (3.8) we use the usual technique for the evaluation of the Mellin-Barnes integral and the residue theory. According to (3.3) and (3.4), poles in (3.1) and (3.2) do not coincide, and all poles a_{ik} in (3.2) are simple. Calculating the residues of the integrand of (3.9) at the simple poles b_l in (3.1) and taking into account the asymptotic formula [8, 1.1(8)]

$$\Gamma(z) = \frac{(-1)^k}{k!(z+k)}[1+O(z+k)] \quad (z \to -k;\ k = 0, 1, 2, \cdots), \tag{3.15}$$

we have

$$I(z) = \sum_{k=0}^{\infty} \operatorname*{Res}_{s=-k} \left[\frac{\Gamma(s) \prod\limits_{i=1}^{p} \Gamma(a_i - \alpha_i s)}{\prod\limits_{j=1}^{q} \Gamma(b_j - \beta_j s)} (-z)^{-s} \right] = \sum_{k=0}^{\infty} \frac{\Gamma(a_i + \alpha_i k)}{\Gamma(b_j + \beta_j k)} \frac{z^k}{k!}. \tag{3.16}$$

Taking (3.9) and (1.1) into account, we deduce the result in (3.8) which completes the proof of the theorem.

Corollary 2.1. *If $\alpha > 0$, $\beta \in \mathbb{C}$ and $z \neq 0$, then the Wright function $\phi(\alpha, \beta; z)$ in* (1.2) *has the integral representation by the Mellin-Barnes integral of the form*:

$$\phi(\alpha, \beta; z) = \frac{1}{2\pi i} \int_{L_{-\infty}} \frac{\Gamma(s)}{\Gamma(\beta - \alpha s)} (-z)^{-s} ds, \tag{3.17}$$

where the path of integration $L_{-\infty}$ separates all poles b_l of (3.1) *to the left.*

Corollary 2.2. *If $\mu > 0$, $\nu \in \mathbb{C}$ and $z \neq 0$, then the Bessel-Maitland function $J_\nu^\mu(z)$ in* (1.3) *has the integral representation by the Mellin-Barnes integral of the form*:

$$J_\nu^\mu(z) = \frac{1}{2\pi i} \int_{L_{-\infty}} \frac{\Gamma(s)}{\Gamma(\nu + 1 - \mu s)} (-z)^{-s} ds, \tag{3.18}$$

where the path of integration $L_{-\infty}$ separates all poles b_l of (3.1) *to the left.*

4. The representation of the generalized Wright function by the Mellin-Barnes integral. Continuation

The representation (1.5) for the generalized Wright function (1.1) was proved in Theorem 2, provided that the contour L is taken to be $L_{-\infty}$ and either of the conditions in (3.5), (3.6) or (3.7) is satisfied. These conditions are conditions on parameters $a_i, b_j \in \mathbb{C}$ and $\alpha_i > 0,\ \beta_j > 0$ ($i = 1,2,\cdots,p;\ j = 1,2,\cdots,q$) of the generalized Wright function ${}_p\Psi_q(z)$. The representation of the form (1.5) stay true for other values of these parameters if we replace the path of integration $L_{-\infty}$ by another contour. In this section we establish the corresponding results for two contours $L_{+\infty}$ and $L_{i\gamma\infty}$ defined as the following: $L = L_{+\infty}$ is a loop situated in a horizontal strip starting at the point $+\infty + i\varphi_1$ and terminating at the point $+\infty + i\varphi_2$ with $-\infty < \varphi_1 < \varphi_2 < \infty$; $L = L_{i\gamma\infty}$ is a contour starting at the point $\gamma - i\infty$ and terminating at the point $\gamma + i\infty$, where $\gamma \in \mathbb{R}$.

First we present the result for the former contour. There holds the following statement, in which Δ, δ and μ are given by (2.1), (2.2) and (2.3), respectively.

THEOREM 3. *Let $a_i, b_j \in \mathbb{C}$ and $\alpha_i > 0,\ \beta_j > 0$ ($i = 1,2,\cdots,p;\ j = 1,2,\cdots,q$) be such that the conditions in* (3.3) *and* (3.4) *are satisfied. Let either of the following conditions holds*:

$$\Delta < -1, \quad z \neq 0; \tag{4.1}$$

$$\Delta = -1, \quad |z| > \delta; \tag{4.2}$$

$$\Delta = -1, \quad |z| = \delta, \quad \Re(\mu) > \frac{1}{2}. \tag{4.3}$$

Then the generalized Wright function ${}_p\Psi_q(z)$ has the integral representation by the Mellin-Barnes integral of the form (1.5) :

$${}_p\Psi_q\left[\begin{array}{c|c} (a_i,\alpha_i)_{1,p} & \\ & z \\ (b_i,\beta_i)_{1,q} & \end{array}\right] = \frac{1}{2\pi i}\int_{L_{+\infty}} \frac{\Gamma(s)\prod_{i=1}^{p}\Gamma(a_i-\alpha_i s)}{\prod_{j=1}^{q}\Gamma(b_j-\beta_j s)}(-z)^{-s}ds, \tag{4.4}$$

where the path of integration $L_{+\infty}$ separates all poles b_l in (3.1) *to the left and all poles a_{ik} in* (3.2) *to the right.*

P r o o f. Let $I(z)$ be given by (3.9)-(3.10). Using the relation (3.11), we deduce the following asymptotic estimate for the function $\mathcal{H}^{1,p}_{p,q+1}(s)$ in the form (3.12):

$$\mathcal{H}^{1,p}_{p,q+1}(\sigma+it) \sim C\left(\frac{e}{t}\right)^{-(\Delta+1)t}\delta^t t^{-[\Re(\mu)+1/2]} \quad (\sigma,t\in\mathbb{R};\ t\to+\infty), \tag{4.5}$$

where

$$C = (2\pi)^{(p+1-q)/2}e^{q-p-\sigma\pi}\frac{\prod_{j=1}^{q}\left[\beta_j^{-\Re(b_j)+1/2}e^{\Re(b_j)-1}\right]\prod_{i=1}^{p}e^{\pi[\sigma\alpha_i+1-\Im(a_i)]}}{\prod_{i=1}^{p}\left[\alpha_i^{-\Re(a_i)+1/2}e^{\Re(a_i)-1}\right]\prod_{j=1}^{q}e^{\pi[\sigma\beta_j+1-\Im(b_j)]}}. \tag{4.6}$$

According to (4.5), (3.14) and the known convergence principles for the improper integrals, the integral in the right hand side of (4.4) is convergent, provided that either of the conditions in (4.1), (4.2) and (4.3) is satisfied. The relation (4.4) is proved similarly to that of (3.8) in Theorem 2, and thus the theorem is proved.

Next result yields the Mellin-Barnes integral representation for the generalized Wright fnction (1.1) on the contour $L_{i\gamma\infty}$.

THEOREM 4. *Let* $a_i, b_j \in \mathbb{C}$ *and* $\alpha_i > 0, \beta_j > 0$ $(i = 1, 2, \cdots, p;\ j = 1, 2, \cdots, q)$ *be such that the conditions in* (3.3) *and* (3.4) *are satisfied and let* $\gamma \in \mathbb{R}$. *Let either of the following conditions holds*:

$$\Delta < 1, \quad |\arg(-z)| < \frac{(1-\Delta)\pi}{2}, \quad z \neq 0; \tag{4.7}$$

$$\Delta = 1, \quad (\Delta + 1)\gamma + \frac{1}{2} < \Re(\mu), \quad \arg(-z) = 0, \quad z \neq 0. \tag{4.8}$$

Then the generalized Wright function ${}_p\Psi_q(z)$ *has the integral representation by the Mellin-Barnes integral of the form* (1.5) :

$${}_p\Psi_q\left[\begin{array}{c|c} (a_i,\alpha_i)_{1,p} & \\ & z \\ (b_i,\beta_i)_{1,q} & \end{array}\right] = \frac{1}{2\pi i}\int_{L_{i\gamma\infty}} \frac{\Gamma(s)\prod_{i=1}^{p}\Gamma(a_i-\alpha_i s)}{\prod_{j=1}^{q}\Gamma(b_j-\beta_j s)}(-z)^{-s}ds, \tag{4.9}$$

where the path of integration $L_{i\gamma\infty}$ *separates all poles* b_l *in* (3.1) *to the left and all poles* a_{ik} *in* (3.2) *to the right.*

P r o o f. As in the proofs of Theorems 2 and 3, we use the notations (3.9) and (3.10). The asymptotic relation

$$\Gamma(x+iy) \sim (2\pi)^{1/2}|x|^{x-1/2}e^{-x-\pi[1-\mathrm{sign}(x)]y/2} \quad (x, y \in \mathbb{R}; \quad |x| \to \infty). \tag{4.10}$$

follows from (2.6). From here we obtain the asymptotic estimate

$$\mathcal{H}^{1,p}_{p,q+1}(t+i\sigma) \sim D|t|^{(\Delta+1)\sigma-\Re(\mu)-1/2}e^{-\pi[|t|(1-\Delta)+\Im(\xi)\mathrm{sign}(t)]/2} \tag{4.11}$$

$$(t, \sigma \in \mathbb{R};\ |t| \to \infty),$$

where

$$D = (2\pi)^{(p+1-q)/2}e^{-\eta}\delta^{\sigma}\frac{\prod_{i=1}^{p}\alpha_i^{\Re(a_i)-1/2}}{\prod_{j=1}^{q}\beta_j^{\Re(b_j)-1/2}}, \tag{4.12}$$

$$\eta = \frac{p-q}{2} + (\Delta+1)\sigma - \Re(\mu), \quad \xi = \mu + \frac{p-q}{2}. \tag{4.13}$$

Thus, taking into account (3.14) and the known convergence principles for the improper integrals, the integral in the right hand side of (4.9) is convergent, provided that either of the conditions in (4.7) and (4.8) is valid. The relation (4.9) is proved similarly to that of (3.8) in Theorem 2, and hence theorem is proved.

Corollary 4.1. *Let $\alpha > 0$, $\beta \in \mathbb{C}$, $z \neq 0$ and $\gamma \in \mathbb{R}$ be such that either of the following conditions holds*:

$$\alpha < 1, \quad |\arg(-z)| < \frac{(1-\alpha)\pi}{2}, \quad z \neq 0; \tag{4.14}$$

$$\alpha = 1, \quad \Re(\beta) > 1 + 2\gamma, \quad \arg(-z) = 0, \quad z \neq 0. \tag{4.15}$$

Then the Wright function $\phi(\alpha, \beta; z)$ in (1.2) *has the integral representation by the Mellin-Barnes integral of the form* (3.17) :

$$\phi(\alpha, \beta; z) = \frac{1}{2\pi i} \int_{L_{i\gamma\infty}} \frac{\Gamma(s)}{\Gamma(\beta - \alpha s)} (-z)^{-s} ds, \tag{4.16}$$

where the path of integration $L_{i\gamma\infty}$ separates all poles b_l of (3.1) *to the left.*

Corollary 4.2. *Let $\mu > 0$, $\nu \in \mathbb{C}$, $z \neq 0$ and $\gamma \in \mathbb{R}$ be such that either of the following conditions holds*:

$$\mu < 1, \quad |\arg(-z)| < \frac{(1-\mu)\pi}{2}, \quad z \neq 0; \tag{4.17}$$

$$\mu = 1, \quad \Re(\nu) > 2\gamma, \quad \arg(-z) = 0, \quad z \neq 0. \tag{4.18}$$

Then the Bessel-Maitland function $J_\nu^\mu(z)$ in (1.3) *has the integral representation by the Mellin-Barnes integral of the form* (4.16) :

$$J_\nu^\mu(z) = \frac{1}{2\pi i} \int_{L_{i\gamma\infty}} \frac{\Gamma(s)}{\Gamma(\nu + 1 - \mu s)} (-z)^{-s} ds, \tag{4.19}$$

where the path of integration $L_{i\gamma\infty}$ separates all poles b_l of (3.1) *to the left.*

Remark 1. The integral representation (4.16) was proved in the paper by Krätzel and Menzer [18], provided that the condition in (4.14) or the condition in (4.15) with $\gamma = 0$ are satisfied.

Remark 2. Wright [35] proved the integral representation for $\phi(\alpha, \beta; z)$ in the form

$$\phi(\alpha, \beta; z) = \frac{1}{2\pi} \int_{-\infty}^{(0+)} t^{-\beta} \exp\left(t + \frac{z}{t^\alpha}\right) dt \quad (\alpha > 0, \quad \beta \in \mathbb{C}), \tag{4.20}$$

where the path of integration starts from $-\infty$ on the real axis, passes round the origin in a counterclockwise direction and returns at $-\infty$; see [9, 18.1(29)]. In [35] and [39] Wright applied the formula (4.20) to establish the asymptotic expansions of $\phi(\alpha, \beta; z)$ for large values of z in the cases $\alpha > 0$ and $-1 < \alpha < 0$, respectively.

Remark 3. Krätzel [17] introduced a modification of (4.20) in the form

$$Z_{\varrho}^{\nu}(x) = \int_0^{\infty} t^{\gamma-1} \exp\left(-t^{\varrho} - \frac{x}{t}\right) dt \quad (x > 0) \tag{4.21}$$

with $\varrho > 0$ and $\gamma \in \mathbb{C}$, established asymptotic estimates of $\phi(\alpha, \beta; z)$ at zero and infinity, and proved convolution theorem and inversion relation for the integral operator K_{ϱ}^{ν} with such a function kernel

$$\left(K_{\varrho}^{\nu} f\right)(x) = \int_0^{\infty} Z_{\varrho}^{\nu}(xt) f(t) dt \quad (x > 0). \tag{4.22}$$

We note that the operational calculus of the integral transform (4.22) was constructed in [31], and compositions of the operator K_{ϱ}^{ν} with the Riemann-Liouville fractional integral and differential operators were proved in [15] with applications to solution of some ordinary differential equations, while the mapping properties such as the boundedness, the range and the representation of the operator K_{ϱ}^{ν} in weighted spaces of r-summable functions were established in [11].

5. The generalized Wright functions as the H-function

Using (3.8), (4.4) and (4.9) we can represent the generalized Wright function (1.1) as a special case of the H-function. This function for integers m, n, p, q such that $0 \leq m \leq q,\ 0 \leq n \leq p$; $a_i,\ b_j \in \mathbb{C}$; $\alpha_i,\ \beta_j \in \mathbb{R}$ $(1 \leq i \leq p,\ 1 \leq j \leq q)$ is defined by

$$H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_i, \alpha_i)_{1,p} \\ \\ (b_j, \beta_j)_{1,q} \end{matrix}\right]$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod\limits_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod\limits_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s)}{\prod\limits_{i=n+1}^{p} \Gamma(a_i + \alpha_i s) \prod\limits_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s)} z^{-s} ds, \tag{5.1}$$

where the contour L is either $L_{-\infty}$, $L_{+\infty}$ or $L_{i\gamma\infty}$, and empty products, if they occur, are taken to be one. As it was indicated in Section 1, the properties of the H-function can be found in the paper by Braaksma [2] and in the books by Erdelyi, etc. [8, Section 1.19], Kilbas and Saigo [14, Chapters 1 and 2], Mathai and Saxena [26, Chapter 2], Prudnikov, Brychkov and Marichev [30, Section 8.3] and Srivastava, Gupta and Goyal [32, Chapter 1].

According to (1.5), the generalized Wright function (1.1) can be represented as a special H-function (5.1) of the form

$$ {}_p\Psi_q \left[\begin{array}{c|c} (a_i, \alpha_i)_{1,p} & \\ & z \\ (b_i, \beta_i)_{1,q} & \end{array} \right] = H^{1,p}_{p,q+1} \left[-z \,\middle|\, \begin{array}{c} (1-a_i, \alpha_i)_{1,p} \\ \\ (0,1), (1-b_j, \beta_j)_{1,q} \end{array} \right]. \tag{5.2} $$

From Theorems 2-4 we deduce the conditions for such a representation.

THEOREM 5. *Let $a_i, b_j \in \mathbb{C}$ and $\alpha_i > 0$, $\beta_j > 0$ $(i = 1, 2, \cdots, p;\ j = 1, 2, \cdots, q)$ be such that the conditions in* (3.3) *and* (3.4) *are satisfied and let $\gamma \in \mathbb{R}$. Let the contour L separates all poles b_l in* (3.1) *to the left and all poles a_{ik} in* (3.2) *to the right. Let one of the following conditions holds*:

(a) $L = L_{-\infty}$ *and either of the conditions in* (3.5), (3.6) *or* (3.7) *is valid.*

(b) $L = L_{+\infty}$ *and either of the conditions in* (4.1), (4.2) *or* (4.3) *is valid.*

(c) $L = L_{i\gamma\infty}$ *and either of the conditions in* (4.7) *or* (4.8) *is valid.*

Then the generalized Wright function (1.1) *is represented as the H-function by* (5.2).

COROLLARY 5.1. *If $\alpha > 0$, $\beta \in \mathbb{C}$ and $z \neq 0$, then the Wright function $\phi(\alpha, \beta; z)$ is represented as the H-function by*

$$ \phi(\alpha, \beta; z) = H^{1,0}_{0,2} \left[-z \,\middle|\, \begin{array}{c} \overline{} \\ \\ (0,1), (1-\beta, \alpha) \end{array} \right], \tag{5.3} $$

where the path of integration $L_{-\infty}$ of the H-function is taken such that it separates all poles b_l of (3.1) *to the left.*

COROLLARY 5.2. *Let $\alpha > 0$, $\beta \in \mathbb{C}$ and $z \neq 0$, then the Wright function $\phi(\alpha, \beta; z)$ is represented as the H-function by* (5.3), *where the contour $L = L_{i\gamma\infty}$ $(\gamma \in \mathbb{R})$ of the H-function is taken such that it separates all poles b_l of* (3.1) *to the left and either of the conditions in* (4.14) *or* (4.15) *are satisfied.*

COROLLARY 5.3. *If $\mu > 0$, $\nu \in \mathbb{C}$ and $z \neq 0$, then the Bessel-Maitland function $J^\mu_\nu(z)$ is represented as the H-function by*

$$ J^\mu_\nu(z) = H^{1,0}_{0,2} \left[-z \,\middle|\, \begin{array}{c} \overline{} \\ \\ (0,1), (-\nu, \mu) \end{array} \right], \tag{5.4} $$

where the path of integration $L_{-\infty}$ of the H-function is taken such that it separates all poles b_l of (3.1) *to the left.*

COROLLARY 5.4. *Let $\mu > 0$, $\nu \in \mathbb{C}$ and $z \neq 0$, then the Wright function $\phi(\alpha, \beta; z)$ is represented as the H-function by* (5.4), *where the contour $L = L_{i\gamma\infty}$ ($\gamma \in \mathbb{R}$) of the H-function is taken such that it separates all poles b_l of* (3.1) *to the left and either of the conditions in* (4.17) *or* (4.18) *are satisfied.*

6. Mittag-Leffler-type functions as special cases of the generalized Wright function and of the H-function

In this section we consider special cases of the generalized Wright function (1.1) which yield the Mittag-Leffler function defined in [6, Chapter III, §1], [7, Chapter 1], [9, Section 18.1]

$$\mathbf{E}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (z, \beta \in \mathbb{C},\ \alpha > 0), \tag{6.1}$$

and some of its generalizations. When $p = 1$, $q = 1$, $a_1 = \alpha_1 = 1$ and $b_1 = \beta$, $\beta_1 = \alpha$, (1.1) coincides with (6.1):

$${}_1\Psi_1 \left[\begin{array}{c|c} (1,1) & \\ & z \\ (\beta, \alpha) & \end{array} \right] = \mathbf{E}_{\alpha,\beta}(z). \tag{6.2}$$

The parameters Δ, δ and μ, given in (2.1)-(2.3), for such a Wright function have the forms:

$$\Delta = \alpha - 1, \quad \delta = |\alpha|^{\alpha}, \quad \mu = \beta - 1. \tag{6.3}$$

Then from Theorems 1, 2, 4 and 5 we deduce the following result.

THEOREM 6. *Let $\alpha > 0$, $\beta \in \mathbb{C}$ and $z \in \mathbb{C}$. There hold the following assertions*:
(a) *The Mittag-Leffler function* (6.1) *is an entire function of z.*
(b) *If $z \neq 0$, then*

$$\begin{aligned} \mathbf{E}_{\alpha,\beta}(z) &= \frac{1}{2\pi i} \int_{L_{-\infty}} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta - \alpha s)} (-z)^{-s} ds \\ &= H^{1,1}_{1,2} \left[-z \,\middle|\, \begin{array}{c} (0,1) \\ \\ (0,1), (1-\beta, \alpha) \end{array} \right], \end{aligned} \tag{6.4}$$

where the path of integration $L_{-\infty}$ separates all poles b_l in (3.1) *to the left.*

(c) *If* $z \neq 0$ *and either* $0 < \alpha < 2$, $|\arg(-z)| < (2-\alpha)\pi/2$ *or* $\alpha = 2$, $\Re(\beta) > 2\gamma + 3/2$ $(\gamma \in \mathbb{R})$, $\arg(-z) = 0$, *then*

$$\mathbf{E}_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_{L_{i\gamma\infty}} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-\alpha s)}(-z)^{-s}ds$$

$$= H^{1,1}_{1,2}\left[-z \,\middle|\, \begin{matrix} (0,1) \\ \\ (0,1),(1-\beta,\alpha) \end{matrix} \right], \tag{6.5}$$

where the path of integration $L_{i\gamma\infty}$ *separates all poles* b_l *in* (3.1) *to the left.*

If $p = 1$, $q = 1$, $a_1 = \rho$, $\alpha_1 = 1$ and $b_1 = \beta$, $\beta_1 = \alpha$ then (1.1) takes the form

$${}_1\Psi_1\left[\begin{matrix} (\rho,1) \\ \\ (\beta,\alpha) \end{matrix} \,\middle|\, z \right] = \Gamma(\rho)E^{\rho}_{\alpha,\beta}(z), \tag{6.6}$$

where the generalized Mittag-Leffler function introduced by Prabhakar [29] (see also [16, (E.32)]) is defined by

$$E^{\rho}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!} \quad (z,\beta,\rho \in \mathbb{C};\ \alpha > 0). \tag{6.7}$$

Here $(\rho)_k$ is the Pochhammer symbol [8, Chapter 1] defined for complex $\rho \in \mathbb{C}$ and $k = 0, 1, 2, \cdots$ by

$$(\rho)_0 = 1, \quad (\rho)_k = \rho(\rho+1)\cdots(\rho+k-1) \quad (k = 1, 2, \cdots). \tag{6.8}$$

Here the parameters Δ, δ and μ have the forms:

$$\Delta = \alpha - 1, \quad \delta = |\alpha|^{\alpha}, \quad \mu = \beta - \rho. \tag{6.9}$$

Then Theorems 1, 2, 4 and 5 yield

Theorem 7. *Let* $\alpha > 0$, $\beta, \rho \in \mathbb{C}$ *and* $z \in \mathbb{C}$. *There hold the following assertions*:

(a) *The generalized Mittag-Leffler function* (6.7) *is an entire function of* z.

(b) *If* $z \neq 0$, *then*

$$\mathbf{E}^{\rho}_{\alpha,\beta}(z) = \frac{1}{2\pi i\Gamma(\rho)}\int_{L_{-\infty}} \frac{\Gamma(s)\Gamma(\rho-s)}{\Gamma(\beta-\alpha s)}(-z)^{-s}ds$$

$$= \frac{1}{\Gamma(\rho)}H^{1,1}_{1,2}\left[-z \,\middle|\, \begin{matrix} (1-\rho,1) \\ \\ (0,1),(1-\beta,\alpha) \end{matrix} \right], \tag{6.10}$$

where the path of integration $L_{-\infty}$ separates all poles b_l in (3.1) *to the left and all poles $\rho + k$ $(k = 0, 1, 2, \cdots)$ to the right.*

(c) *If $z \neq 0$ and either $0 < \alpha < 2$, $|\arg(-z)| < (2-\alpha)\pi/2$ or $\alpha = 2$, $\Re(\beta) > 2\gamma + \Re(\rho) + 1/2$ and $\arg(-z) = 0$, then*

$$\mathbf{E}^{\rho}_{\alpha,\beta}(z) = \frac{1}{2\pi i \Gamma(\rho)} \int_{L_{i\gamma\infty}} \frac{\Gamma(s)\Gamma(\rho - s)}{\Gamma(\beta - \alpha s)} (-z)^{-s} ds$$

$$= \frac{1}{\Gamma(\rho)} H^{1,1}_{1,2}\left[-z \,\middle|\, \begin{matrix} (1-\rho, 1) \\ \\ (0,1), (1-\beta, \alpha) \end{matrix} \right], \tag{6.11}$$

where the path of integration $L_{i\gamma\infty}$ $(\gamma \in \mathbb{R})$ separates all poles b_l in (3.1) *to the left and all poles $\rho + k$ $(k = 0, 1, 2, \cdots)$ to the right.*

It is also used the following modification of the generalized Mittag-Leffler function (6.7):

$$E^m_\rho(z;\omega) = \sum_{k=0}^{\infty} \frac{\Gamma(m+1+k)}{\Gamma(\omega + k/\rho)} \frac{z^k}{k!} \tag{6.12}$$

$$(m \in \mathbf{N}_0 = \{0, 1, 2, \cdots\};\ \rho > 0,\ z, \omega \in \mathbb{C}),$$

introduced by Imanaliev and Weber [13], see also [16, (E.32)]. For such a function the relations (6.6) and (6.9) take the forms

$${}_1\Psi_1 \left[\begin{matrix} (m+1, 1) \\ \\ (\omega, 1/\rho) \end{matrix} \,\middle|\, z \right] = E^m_\rho(z;\omega), \tag{6.13}$$

and

$$\Delta = \frac{1}{\rho} - 1, \quad \delta = |\rho|^{-1/\rho}, \quad \mu = \omega - m - 1, \tag{6.14}$$

while Theorem 7 transfers to the following result.

Theorem 8. *Let $\rho > 0$, $m \in \mathbf{N}_0$, $\omega \in \mathbb{C}$ and $z \in \mathbb{C}$. There hold the following assertions*:

(a) *The generalized Mittag-Leffler function* (6.12) *is an entire function of z.*

(b) *If $z \neq 0$, then*

$$E^m_\rho(z;\omega) = \frac{1}{2\pi i} \int_{L_{-\infty}} \frac{\Gamma(s)\Gamma(m+1-s)}{\Gamma(\omega - s/\rho)} (-z)^{-s} ds$$

$$= H^{1,1}_{1,2}\left[-z \,\middle|\, \begin{matrix} (-m, 1) \\ \\ (0,1), (1-\omega, 1/\rho) \end{matrix} \right], \tag{6.15}$$

where the path of integration $L_{-\infty}$ separates all poles b_l in (3.1) *to the left and all poles $m+k+1$ $(k=0,1,2,\cdots)$ to the right.*

(c) *If $z \neq 0$ and either $\rho > 1/2$, $|\arg(-z)| < (2\rho-1)\pi/(2\rho)$ or $\rho = 1/2$, $\Re(\omega) > 2\gamma + m + 3/2$ $(\gamma \in \mathbb{R})$, $\arg(-z) = 0$, then*

$$E_\rho^m(z;\omega) = \frac{1}{2\pi i}\int_{L_{i\gamma\infty}} \frac{\Gamma(s)\Gamma(m+1-s)}{\Gamma(\omega - s/\rho)}(-z)^{-s}ds$$

$$= H_{1,2}^{1,1}\left[-z \,\middle|\, \begin{matrix}(-m,1)\\ \\ (0,1),(1-\omega,1/\rho)\end{matrix}\right], \tag{6.16}$$

where the path of integration $L_{i\gamma\infty}$ separates all poles b_l in (3.1) *to the left and all poles $m+k+1$ $(k=0,1,2,\cdots)$ to the right.*

For $p=1$, $q=2$ and $a_1 = \alpha_1 = 1$, $b_i = \beta_i$, $\beta_i = \alpha_i$ $(i=1,2)$

$${}_1\Psi_2\left[\begin{matrix}(1,1)\\ \\ (\beta_1,\alpha_1),(\beta_2,\alpha_2)\end{matrix}\,\middle|\, z\right] = \Phi_{\alpha_1,\alpha_2}(z;\beta_1,\beta_2), \tag{6.17}$$

where the function $\Phi_{\alpha_1,\alpha_2}(z;\beta_1,\beta_2)$, defined by

$$\Phi_{\alpha_1,\alpha_2}(z;\beta_1,\beta_2) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha_1 k+\beta_1)\Gamma(\alpha_2 k+\beta_2)}z^k \tag{6.18}$$

$$(z,\beta_1,\beta_2 \in \mathbb{C},\ \alpha_1,\alpha_2 > 0),$$

was introduced by Djzrbashian [5], see also [12, (4.9)] and [16, (E.33)]. The parameters Δ, δ and μ have the forms:

$$\Delta = \alpha_1 + \alpha_2 - 1, \quad \delta = |\alpha_1|^{\alpha_1}|\alpha_2|^{\alpha_2}, \quad \mu = \beta_1 + \beta_2 - \frac{3}{2}, \tag{6.19}$$

and from Theorems 1, 2, 4 and 5 we obtain

Theorem 9. *Let $\alpha_i \in \mathbb{R}$ $(\alpha_i \neq 0)$, $\beta_i \in \mathbb{C}$ $(i=1,2)$ and $z \in \mathbb{C}$. There hold the following assertions*:

(a) *If $\alpha_1 + \alpha_2 > 0$, then the generalized Mittag-Leffler function* (6.18) *is an entire function of z.*

(b) *If $\alpha_1 + \alpha_2 = 0$ and either $|z| < |\alpha_1|^{\alpha_1}|\alpha_2|^{\alpha_2}$ or $|z| = |\alpha_1|^{\alpha_1}|\alpha_2|^{\alpha_2}$, $\Re(\beta_1 + \beta_2) > 2$, then the series in (6.17) is absolutely convergent.*

(c) *If $\alpha_i > 0$ $(i=1,2)$ and $z \neq 0$, then*

$$\Phi_{\alpha_1,\alpha_2}(z;\beta_1,\beta_2) = \frac{1}{2\pi i}\int_{L_{-\infty}} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta_1-\alpha_1 s)\Gamma(\beta_2-\alpha_2 s)}(-z)^{-s}ds$$

$$= H_{1,3}^{1,1}\left[-z \,\middle|\, \begin{matrix} (0,1) \\ \\ (0,1),(1-\beta_1,\alpha_1),(1-\beta_2,\alpha_2) \end{matrix}\right], \tag{6.20}$$

where the path of integration $L_{-\infty}$ separates all poles b_l in (3.1) *to the left and all poles $k+1$ $(k=0,1,2,\cdots)$ to the right.*

(d) *If $\alpha_i > 0$ $(i=1,2)$, $z \neq 0$ and either $\alpha_1+\alpha_2 < 2$, $|\arg(-z)| < (2-\alpha_1+\alpha_2)\pi/2$ or $\alpha_1+\alpha_2 = 2$, $\Re(\beta_1+\beta_2) > 2\gamma+2$ $(\gamma \in \mathbb{R})$, $\arg(-z)=0$, then*

$$\Phi_{\alpha_1,\alpha_2}(z;\beta_1,\beta_2) = \frac{1}{2\pi i}\int_{L_{i\gamma\infty}} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta_1-\alpha_1 s)\Gamma(\beta_2-\alpha_2 s)}(-z)^{-s}ds$$

$$= H_{1,3}^{1,1}\left[-z \,\middle|\, \begin{matrix} (0,1) \\ \\ (0,1),(1-\beta_1,\alpha_1),(1-\beta_2,\alpha_2) \end{matrix}\right], \tag{6.21}$$

where the path of integration $L_{i\gamma\infty}$ separates all poles b_l in (3.1) *to the left and all poles $k+1$ $(k=0,1,2,\cdots)$ to the right.*

When $p=1$, $q=m$, $a_1=\rho$, $\alpha_1=1$ and $b_j=\beta j$, $\beta_j=\alpha_j$ $(j=1,\cdots,m)$, then

$${}_1\Psi_m\left[\begin{matrix} (\rho,1) \\ \\ (\beta_1,\alpha_1),\cdots,(\beta_m,\alpha_m) \end{matrix}\,\middle|\, z\right] = \Gamma(\rho)E_\rho[(\beta,\alpha)_m;z], \tag{6.22}$$

where the generalized Mittag-Leffler function $E_\rho[((\beta,\alpha)_m;z]$ is defined by

$$E_\rho[(\beta,\alpha)_m;z] \equiv E_\rho\left[(\beta_1,\alpha_1),\cdots,(\beta_m,\alpha_m);z\right]$$

$$= \sum_{k=0}^{\infty} \frac{(\rho)_k}{\prod_{j=1}^{m}\Gamma(\alpha_j k+\beta_j)}\frac{z^k}{k!}, \tag{6.23}$$

see, for example, Al-Bassam and Luchko in [1, (63)].

Here the parameters Δ, δ and μ are given by

$$\Delta = \sum_{j=1}^{m}\alpha_j - 1, \quad \delta = \prod_{j=1}^{m}|\alpha_j|^{\alpha_j}, \quad \mu = \sum_{j=1}^{m}\beta_j - \rho + \frac{1-m}{2}, \tag{6.24}$$

and from Theorems 1, 2, 4 and 5 we deduce the result.

Theorem 10. *Let $\alpha_i \in \mathbb{R}$ $(\alpha_i \neq 0)$, $\beta_i \in \mathbb{C}$ $(i=1,2,\cdots,m)$ and $z \in \mathbb{C}$. There hold the following assertions*:

(a) *If $\sum_{j=1}^{m}\alpha_j > 0$, then the generalized Mittag-Leffler function* (6.23) *is an entire function of z.*

(b) *If* $\sum_{j=1}^{m}\alpha_j = 0$ *and either* $|z| < \prod_{j=1}^{m}|\alpha_j|^{\alpha_j}$ *or* $|z| = \prod_{j=1}^{m}|\alpha_j|^{\alpha_j}$, $\sum_{j=1}^{m}\Re(\beta_j) > \Re(\rho) + m/2$, *then the series in* (6.23) *is absolutely convergent.*

(c) *If* $\alpha_i > 0$ $(i = 1, 2)$ *and* $z \neq 0$, *then*

$$E_\rho[(\beta,\alpha)_m; z] = \frac{1}{2\pi i}\int_{L_{-\infty}} \frac{\Gamma(s)\Gamma(\rho - s)}{\prod_{j=1}^{m}\Gamma(\beta_j - \alpha_j s)}(-z)^{-s}ds =$$

$$= H^{1,1}_{1,m+1}\left[-z \,\middle|\, \begin{array}{l} (0,1) \\ \\ (0,1),(1-\beta_1,\alpha_1),\cdots,(1-\beta_m,\alpha_m) \end{array}\right], \tag{6.25}$$

where the path of integration $L_{-\infty}$ *separates all poles* b_l *in* (3.1) *to the left and all poles* $\rho + k$ $(k = 0, 1, 2, \cdots)$ *to the right.*

(d) *If* $\alpha_i > 0$ $(i = 1, 2)$, $z \neq 0$ *and either* $\sum_{j=1}^{m}\alpha_j < 2$, $|\arg(-z)| < \left(2 - \sum_{j=1}^{m}\alpha_j\right)\frac{\pi}{2}$ *or* $\sum_{j=1}^{m}\alpha_j = 2$, $\sum_{j=1}^{m}\Re(\beta_j) > 2\gamma + \Re(\rho) + \frac{m}{2}$ $(\gamma \in \mathbb{R})$, $\arg(-z) = 0$, *then*

$$E_\rho[((\beta,\alpha)_m; z] = \frac{1}{2\pi i}\int_{L_{i\gamma\infty}} \frac{\Gamma(s)\Gamma(1 - s)}{\prod_{j=1}^{m}\Gamma(\beta_j - \alpha_j s)}(-z)^{-s}ds$$

$$= H^{1,1}_{1,m+1}\left[-z \,\middle|\, \begin{array}{l} (0,1) \\ \\ (0,1),(1-\beta_1,\alpha_1),\cdots,(1-\beta_m,\alpha_m) \end{array}\right], \tag{6.26}$$

where the path of integration $L_{i\gamma\infty}$ *separates all poles* b_l *in* (3.1) *to the left and all poles* $\rho + k$ $(k = 0, 1, 2, \cdots)$ *to the right.*

Remark 4. The results in Theorems 6(a) and 7(a) are well known; see for example, [9, Section 18.1] and [29].

Remark 5. The relation (6.4) was indicated in [30, 8.4.51(7)].

Remark 6. Theorems 1-5 can be applied to deduce the corresponding results for the generalized hypergeometric function ${}_pF_q[a_1, \cdots, a_p; b_1, \cdots, b_q; z]$ defined by the generalized hypergeometric series [8, Section 4.1]

$$
{}_pF_q[a_1,\cdots,a_p;b_1,\cdots,b_q;z]=\sum_{k=0}^{\infty}\frac{\prod\limits_{i=1}^{p}(a_i)_k}{\prod\limits_{j=1}^{q}(b_j)_k}, \tag{6.27}
$$

which is connected with generalized Wright function (1.1) with $\alpha_i = \beta_j = 1$ $(i = 1,\cdots,p;\ j = 1,\cdots,q)$ by

$$
{}_p\Psi_q\left[\begin{array}{c|c}(a_i,1)_{1,p} & \\ & z\\ (b_i,1)_{1,q} & \end{array}\right]=\frac{\prod\limits_{i=1}^{p}\Gamma(a_i)}{\prod\limits_{j=1}^{q}\Gamma(b_j)}\,{}_pF_q[a_1,\cdots,a_p;b_1,\cdots,b_q;z] \tag{6.28}
$$

with $a_i \neq -l$ $(i = 1,2,\cdots,n;\ l = 0,1,2,\cdots)$.

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